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# Engineering Notes

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## Eigenstructure Assignment Using Inverse Eigenvalue Methods

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### Introduction

THIS Note points out the relationship between eigenstructure assignment methods for second-order mechanical systems and the inverse eigenvalue problem for second-order matrix polynomials. The inverse method for matrix polynomials is then used to derive a new eigenstructure assignment algorithm using alternate matrix manipulations which incorporate left eigenvectors, perhaps providing insight. Eigenstructure assignment methods have received substantial theoretical development as well as practical applications. Andry et al.[1] provided both a literature survey (prior to 1983) and illuminating results on applications of eigenstructure assignment methodology to control aerospace systems and mechanical systems.[2] Mechanical systems are those which are naturally represented in the form

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = F(t) \tag{1}$$

where $M$, $D$, and $K$ are $n \times n$, symmetric, real valued, positive definite (or semidefinite) matrices representing a system mass, damping, and stiffness matrix, respectively, $x$ is an $n \times 1$ vector of displacements with time derivatives $\dot{x}$ (the velocity vector) and $\ddot{x}$ (the acceleration vector).

The objective of eigenstructure assignment is to develop a control law to apply to the right-hand side of Eq. (1), based on measurement, that will move the eigenvalues and eigenvectors (together referred to as eigenstructure) to more desirable values. The closed-loop response of a system is shaped by the system's natural frequencies, mode shapes, and damping ratios contained in the system eigenstructure. Thus, eigenstructure assignment can alter the response of a system by changing the system's modal characteristics. The approach presented here makes use of the matrix results associated with inverse eigenvalue problems.[3] The goal of this Note is to connect the topic of inverse eigenvalue problems with that of eigenstructure assignment. In so doing, a new method of computing a feedback law for eigenstructure assignment is derived for the full state feedback case.

The eigenvalue problem associated with Eq. (1) becomes (after multiplying by $M^{-1}$)

$$L(\lambda)x = \left(\lambda^2 I + \lambda H_2 + H_3\right)x = 0, \quad x \neq 0 \tag{2}$$

where the complex scalar $\lambda$ is called an eigenvalue and $x$ is now used to denote a right eigenvector, a possibly complex but never zero vector of constants. The matrix polynomial $L(\lambda)$ is referred to as a lambda matrix. Because $L(\lambda)$ is $n \times n$ and of second order, there are $2n$ values of $\lambda$ and of $x$, each of which is denoted by $\lambda_i$ and $x_i$.

Equation (1) is stated in the standard physical coordinate system of dimension $n$, often referred to as second-order form. Two other formulations are used as mathematical models of vibrating structures: the standard state space formulation common to control theory and a first-order formulation in pencil form, both in $2n$ dimensions. Consideration of these various forms allow for the development of a solution to the inverse eigenvalue problem for systems with rigid body modes, which are in turn applied to the eigenstructure assignment problem.

### Inverse Eigenvalue Problem

The matrix polynomial of Eq. (2) also has left eigenvectors denoted $y_i \neq 0$, satisfying

$$y_i^T L(\lambda_i) = 0 \tag{3}$$

The inverse eigenvalue problem for the system of Eq. (2) is stated in terms of the matrix $\Lambda$, defined as the diagonal matrix ($2n \times 2n$) of elements $\lambda_i$, i.e., $\Lambda = \mathrm{diag}(\lambda_i)$, and in terms of the right and left matrices of mode shapes defined by

$$X = [x_1\ x_2 \cdots x_{2n}] \tag{4}$$

$$Y = [y_1\ y_2 \cdots y_{2n}] \tag{5}$$

where $X$ and $Y$ are $n \times 2n$. Simply stated, the inverse eigenvalue problem is to calculate the coefficient matrices $H_2$ and $H_3$ in terms of $X$, $\Lambda$, and $Y$. This problem can be solved for the rigid-body mode case that some eigenvalues are zero by comparing three matrix representations of the eigenvalue problem. Let

$$A = \begin{pmatrix} 0 & I \\ -H_3 & -H_2 \end{pmatrix} \tag{6}$$

denote the standard $2n \times 2n$ state matrix. Then the eigenvalues of $A$ are the solutions of $Aq_i = \lambda_i q_i, q_i \neq 0$ where $q_i$ are the right eigenvectors of $A$ and $\lambda_i$ are, of course, the eigenvalues of $L(\lambda)$. These $2n$ eigenvectors can be arranged into a matrix $Q$ where columns are the $2n \times 1$ eigenvectors $q_i$. Furthermore;

$$Q = \begin{bmatrix} X \\ X\Lambda \end{bmatrix} \tag{7}$$

The state matrix $A$ can then be written as $A = Q\Lambda Q^{-1}$. However directly inverting $Q$ involves $\Lambda^{-1}$, and it is thus desired to calculate $Q^{-1}$ without actually computing an inverse so that $\lambda_i = 0$ for some index may be considered. This can be accomplished by examining the matrix pencil or generalized eigenvalue problem; $Pu = \lambda Nu$, $u \neq 0$, where

$$N = \begin{bmatrix} -H_2 & I \\ I & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} H_3 & I \\ I & 0 \end{bmatrix} \tag{8}$$

It is shown in Ref. 3 that $Q^{-1}$ can be replaced by $Q^{-1} = V^T N$ so that the matrix $A$ can be then written as

$$A = Q\Lambda V^T N \tag{9}$$

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Multiplying out the right side of Eq. (9), comparing the $n \times n$ subblocks and performing some simple algebraic manipulations yields

$$H_2 = -X\Lambda^2 Y^T \tag{10}$$

$$H_3 = H_2^2 - X\Lambda^3 Y^T \tag{11}$$

$$I = X\Lambda Y^T \tag{12}$$

Equations (10) and (11) along with the normalization condition of Eq. (12) provide a solution to the inverse eigenvalue problem for second-order systems containing zero (or very small) eigenvalues. Thus, a system with rigid-body modes or structures with very low frequencies can be constructed by use of this particular set of inverse formulas.

## Eigenstructure Assignment Problem

In second-order form the control problem for the system of Eq. (1) becomes

$$\ddot{x} + M^{-1}D\dot{x} + M^{-1}Kx = BGC_0x + BGC_1\dot{x} \tag{13}$$

where $B$ is the $n \times r$ control input matrix (actually of the form $M^{-1}B_0$), $G$ is the $r \times m$ gain matrix, $C_0$ is the $m \times n$ input matrix of position measurements, and $C_1$ is the $m \times n$ input matrix of velocity measurements. Here, $r$ is the number of actuators, and $m$ is the number of sensors. Equation (13) can also be written in closed-loop form as

$$\ddot{x} + \left(M^{-1}D - BGC_1\right)\dot{x} + \left(M^{-1}K - BGC_0\right)x = 0 \tag{14}$$

The goal of the eigenstructure assignment problem[1,2] is, given the form of Eq. (14), the input/output matrices $B$, $C_0$, and $C_1$, a desired set of eigenvalues $\lambda_r$, and desired set of eigenvectors $x_i$, $i = 1, 2, \ldots, k \leq \max(m, r)$, find the gain matrix $G$ such that the eigenvalues and eigenvectors associated with Eq. (14) contain these desired eigenvalues and eigenvectors.

The solution to this problem provided in Ref. 1 is stated in terms of the state matrix $A$ by first transforming the closed-loop system via the relationship $x = T\tilde{x}$, where $T = [B\ R]$, where $R$ is a random matrix chosen to make $T^{-1}$ exist and be well conditioned. Once transformed, the gain matrix is calculated by

$$G = B^{\dagger}\left[Z_1 + M^{-1}DX_1 + M^{-1}KZ_1\right][C_1X_1 + C_0Z_1]^{-1} \tag{15}$$

where $X_1$ is the $n \times r$ matrix with columns matching those of the desired eigenvectors and $Z_i = [\lambda_1 x_1\ \lambda_2 x_2 \cdots \lambda_r x_r]$. Here $\dagger$ denotes a generalized inverse for the case that $r < m$ and the inverse of $C_1X_1 + C_0Z_1$ is assumed to exist as long as the measurements taken have significant impact on the desired eigenvalues. In practice, this condition is used to provide a check of the reasonableness of the designers choice of mode shapes.[1]

## Comments on Mode Shape Selection

The mode shape choice is also limited by the dynamics of the original structure. For instance it would not be feasible to try to assign a cosine mode shape to the first mode of a cantilevered beam. Mathematically, the idea of "reasonable" choices of mode shapes can be discussed by considering a rearrangement of Eq. (14) in eigenvalue form (i.e., with $x \to x_i$, $\dot{x} \to \lambda_i x_i$ and $\ddot{x} \to \lambda_i^2 x_i$). Let $x_i^d$ denote a particular choice of a desired mode shape and let $\lambda_i$ be the corresponding choice of eigenvalue. Then Eq. (14) can be written as

$$x_i^d = \left\{\left[\lambda_i^2 I + \lambda_i M^{-1}D + M^{-1}K\right]^{-1} B\right\} m_i \tag{16}$$

where $m_i = G[C_0\ \lambda_i C_1]x_i^d$, is an $r \times 1$ vector. Note the inverse used in Eq. (16) exists because $\lambda_i$ is not an eigenvalue of the original system. Next, denote the matrix in brackets by $L_i = [\lambda_i^2 I + \lambda_i M^{-1}D + M^{-1}K]^{-1}B$, and note that Eq. (16) states that the desired mode shape must lie in the space spanned by the columns of $L_i$. In particular, not just any eigenvector can be assigned to a given system (consistent with the physical notion mentioned earlier). Following the state space derivation of Ref. 1 an "achievable" mode shape $x_i^a$ can be calculated from a desired mode shape $x_i^d$ by finding the projection of $x_i^d$ onto the aforementioned subspace. A simple derivative calculation yields

$$x_i^a = L_i\left(L_i^T L_i\right)L_i^T x_i^d \tag{17}$$

Once $x_i^a$ is calculated, Eq. (15) can be used to compute the required gain matrix. Here, the desired eigenstructure will be achieved by the closed-loop system provided full state feedback is used ($m = r = n$). Here, only full state feedback is considered. References 1, 4, and 5 discuss methods and problems of output feedback.

## Eigenstructure Assignment by Inverse Formulas

To adapt the inverse eigenvalue problem to the eigenstructure assignment problem, partition the matrix of eigenvalues into two groups. Let $\Lambda_1 = \text{diag}(\lambda_1\ \lambda_2 \cdots \lambda_r)$ represent the $r$ desired eigenvalues and $\Lambda_2 = (\lambda_{r+1}\ \lambda_{r+2} \cdots \lambda_{2n})$ denote the remaining open-loop eigenvalues which will not be changed so that

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \tag{18}$$

Likewise, partition both the left and right matrices of eigenvectors into

$$X = [X_1 \quad X_2] \qquad \text{and} \qquad Y = [Y_1 \quad Y_2] \tag{19}$$

where $X_1$ are assignable right mode shapes and $Y_1$ are the corresponding left eigenvectors. Substitution of these partitioned forms into the inverse eigenvalue formulas of Eqs. (10–12) yields

$$H_2 = -\left(X_1\Lambda_1^2 Y_1^T + X_2\Lambda_2^2 Y_2^T\right) \tag{20}$$

$$H_3 = H_2^2 - \left(X_1\Lambda_1^3 Y_1^T + X_2\Lambda_2^3 Y_2^T\right) \tag{21}$$

$$I = X_1\Lambda Y_1^T + X_2\Lambda_2 Y_2^T \tag{22}$$

Substitution of these partitioned forms into the feedback control formulation yields

$$BGC_0 = M^{-1}K - H_3 = M^{-1}K - H_2^2 + X_1\Lambda_1^3 Y_1^T + X_2\Lambda_2^3 Y_2^T \tag{23}$$

from the coefficient of the position vector and

$$BGC_1 = M^{-1}D - H_2 = M^{-1}D + X_1\Lambda_1^2 Y_1^T + X_2\Lambda_2^2 Y_2^T \tag{24}$$

from the coefficient of the velocity term. Denoting the right-hand side of Eq. (23) as $\Delta H_3$ and the right-hand side of Eq. (24) as $\Delta H_2$, yields $\Delta H_3 = -B_0GC_0$ and $\Delta H_2 = -B_0GC_1$, which can be written together in partitioned form as

$$BGC = [-\Delta H_3 - \Delta H_2] \tag{25}$$

where $C$ is the $m \times 2n$ matrix $C = [C_0\ C_1]$. This last expression represents an alternative formula for the selection of gains in the eigenstructure assignment problem provided by the inverse eigenvalue solution.

## Comparison

For the full state feedback case ($r = n$), Eq. (25) yields the $n \times 2n$ gain matrix

$$G = B^{-1}[-\Delta H_3 - \Delta H_2]C^{\dagger} \tag{26}$$

By contrast, the standard eigenstructure assignment method of Ref. 1 yields the $n \times 2n$ gain matrix given by Eq. (15). The major differences between these two gain calculations, beyond their obvious disimilar algebraic form, is that the Andry et al. method requires the transformation $T$ and its inverse whereas the approach proposed here does not. In addition, the inverse approach uses the left-eigenvector information which is not required (or taken advantage of) in the standard approach. Computation of the gain matrix $G$ for a given set of desired eigenvalues, eigenvectors, $B$ and $C$ have produced the same value of the gain matrix using either method (even in the case of $r$, $m < n$) in several numerical examples. This should not be to surprising as Moore[6] has shown in the state space that, if the rank $B = r$, which it does here, the matrix $GC$ is unique for a given set of eigenvalues. The gain matrix of the inverse method given in Eq. (26) and that of the standard approach of Eq. (15) can also be derived in the state-space-context as given in Refs. 1 and 7. Repeated numerical examples using MATLAB also seem to indicate that the gain matrix computed using Eq. (26) is computed in about 1/3 the time (about 0.134 s compared to 0.43 s for 4 DOF). This is largely due to calculating $T^{-1}$ in the approach of Ref. 1. MATLAB codes for both gains are available from the author,[7] as are examples of low-order systems.

## Conclusion

An eigenstructure assignment method has been presented based on using inverse eigenvalue theory. The resulting gain matrix does not resemble that obtained by conventional eigenstructure assignment but appears to produce the same numerical values for several different example problems, for less computational effort, when programmed in MATLAB. In addition, the proposed method allows both the open-loop structure and the closed-loop system to have rigid-body modes or very low frequencies often useful in aerospace structures. The remaining modes are stable because of the partitioning offered by the inverse eigenvalue approach when used with full state feedback. Often, full state feedback is looked upon with destain as being impractical. However, recent developments in smart structures technology have rendered full state feedback feasable.[8] The method proposed here allows specific eigenvalues to be changed, leaving the others unchanged.

The condition provided in Eq. (16) for physical mode shapes of mechanical systems should provide insight for those interested in control system design of mechanical systems where the assigned eigenvector is directly related to the physical vibration mode shape. Reference 1 should be consulted for an excellent physical example of flight control in state space. The result presented here makes a strong connection between the seemingly independent fields of inverse eigenvalue problems and that of eigenstructure assignment. Perhaps others can make more fruitful connections.

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# Variance-Based Sensor Placement for Modal Identification of Structures


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## Introduction

In planning a structural identification test one is faced with a problem of where to place a limited number of sensors to maximize the accuracy of the experiments. Such considerations become more important when the sensors cannot be easily moved as in large space structures or when they are to be used for health monitoring.

Various methods for sensor and actuator positioning for control schemes have been developed, and examples are given in Refs. 1 and 2. In particular, Ref. 1 uses a backward elimination approach to pick locations for a linear quadratic regulator (LQR) control scheme. Methods for sensor placement in structural identification are contained in Refs. 3–5. Also within the structural identification field, Kammer[6] formulated and evaluated an effective independence (EfI) approach for the ranking of sensors which is also a backward elimination approach. The motivation was to pick sensor locations to render the resulting columns of the modal matrix as independent as possible. This was accomplished indirectly by considering the solution of a linear equation in which the modal matrix is the coefficient matrix. The accuracy of the resulting solution, measured by the size of the covariance matrix, is related to the linear independence of the modal matrix. A concise and informative derivation of this method is given in Ref. 7. Whereas Ref. 6 measured the size of the covariance matrix by its determinant, an alternative standard measure on the covariance matrix is its trace.[8] This measure has a direct interpretation as the sum of the variances of the estimated parameters. As shown herein, it is also closely related to the linear independence of the columns of the modal matrix through a condition number defined on the Frobenius norm. In this Note the backward elimination and the sequential replacement approaches are applied to the sensor placement problem using the covariance trace criterion. An extension to a condition number of the modal matrix is also given.

## Method 1: Sensor Location to Minimize Variance

It is assumed that an approximate set of $N_t$ target modes are available typically from a finite element analysis of the test structure. The mode shapes are defined at $l$ locations on the structure. The response of the structure for the target modes may be written as

$$y(t) = \Phi_l \eta(t) + w(t) \tag{1}$$

where $y(t)$ is an $l \times N_t$ vector of responses, and $\Phi_l$ is an $l \times N_t$ modal matrix of target modes. The subscript notation for this matrix refers to the number of locations or rows. Here, $\eta(t)$ is the $N_t \times 1$ vector of modal responses (including the modal participation factors) and $w(t)$ is the $l \times 1$ vector of measurement noise. The correlation matrix of this random vector is

$$E[w(t)w(t-\tau)^T] = I\delta(t-\tau) \tag{2}$$

where $E$ is the expectation operator, and $\delta$ is the Kronecker delta function. The best linear unbiased estimate of $\eta(t)$, which is simply the least squares estimate, is

$$\hat{\eta}(t) = \left[\Phi_l^T \Phi_l\right]^{-1} \Phi_l^T y(t) \tag{3}$$

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